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ANALYTICAL SOLUTION OF THE MAGNETIC FIELD EQUATION IN TOROIDAL COORDINATES

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Маълум $\nabla\times\nabla\times\vec{A}=\vec{j}$ кўринишдаги магнит майдон тенгламаси айлана ток майдони учун ўзгарувчиларни бўлиш усули ёрдамида тороидал координатали ташқи дифференциал шакл доирасида ечими топган. Майдон манбаси сифатида ўзгармас токли айлана ҳхалқа кўрилган. Кўрилаётган майдон манбадан ҳар томонлама эркин бўлганлиги учун, айланадан ташқари, тенгламанинг ечилиши кифоя. Ротор майдонини ҳисоблашни осонлаштириш учун вектор потенциали ягона A_φ нолинчи ташкил этувчигли $\alpha=A_i dx^i$ ко-векторга алмаштирилди, ягона нолга тенг бўлмаган компонентага эга бўлган. Майдон тенгламаси $d^* d\alpha=0$ кўринишда тасвирланган. Ҳалқадаги ушбу доимий учун ток потенициали ко-вектори ташкил этувчилар учун содда ифода олинди. Бу ечим асосида куч чизиқларининг рақамли кўриниши қурилган.

Уравнение магнитного поля, известное в виде $\nabla\times\nabla\times\vec{A}=\vec{j}$, решено для поля кругового токового кольца в рамках исчисления внешних дифференциальных форм в тороидальных координатах методом разделения переменных. В качестве источника поля рассматривалось круговое кольцо, по которому течет постоянный ток. Поскольку рассматриваемое поле свободно от источника везде, кроме самого кольца, достаточно было решить однородное уравнение. Для упрощения вычисления ротора поля векторный потенциал был заменен на ко-векторный $\alpha=A_i dx^i$, имеющий единственную ненулевую компоненту A_φ. Уравнение поля было представлено в виде $d^* d\alpha=0$. Получено простое выражение для этой компоненты ко-векторного потенциала постоянного тока в кольце. На основе этого решения численно построены силовые линии.

The magnetic field equation known in the form $\nabla\times\nabla\times\vec{A}=\vec{j}$ is solved in the framework of exterior differential forms in toroidal coordinates for the field of circular current loop by the method of variables separation. The source is assumed to be circular loop in form, which carries stationary current. Since the field is source free everywhere but the loop itself, the uniform equation was solved. To simplify completing the curl operation, the vector potential A is replaced with co-vector one $\alpha=A_i dx^i$ which in toroidal coordinates has single non-zero component A_φ. The equation was represented in terms of exterior differential forms as $d^* d\alpha=0$. A simple closed form expression for this component of the co-vector potential produced by direct current carried by the loop, is presented. Lines of force of the field are obtained from the expression and built numerically.

I. INTRODUCTION

In his monograph "Classical Electrodynamics", J.D. Jackson [1] wrote: The basic differential laws of magnetostatics are given by

$$\nabla\times\vec{B}=(4\pi/c)\vec{j}, \quad \nabla\vec{j}=0. \tag{1}$$

The problem is how to solve them. This problem is of the same nature as numerous classical ones whose solutions form the subject of mathematical physics as it is exposed in standard texts like [2]. The only difference between the equation (1) and a typical equation of mathematical physics is that the earlier is not vectorial. Classical methods of mathematical physics allow one to reduce scalar equations to ordinary differential ones by the method of variables separation, but they tell nothing about non-scalar ones like this one. Moreover, the curl operation encountered here is not known to everyone in its general form which could be applied in an arbitrary coordinate system.

In fact, the only approach applied to the equation (1), is the method of Green functions [3-8]. At the same time, if the source possesses axial symmetry, the task always reduces to contructing a purely toroidal vector field which is either the vector potential or the strength, which in an appropriate coordinate system has single non-zero component, say A_φ, where φ is azimuthal angle as one of coordinates. Therefore the task reduces to constructing a single function from a partial differential equation which is very similar to those solved in scalar theories. In this work we derive explicit form of the partial differential equation for the φ-component of the vector potential solve it by the method of variables separation and show that the solution obtained describes the field of circular current loop.

II. THE FIELD EQUATION IN TOROIDAL COORDINATES

The first task is to derive explicit form of the partial differential equation for the φ-component of the desired vector potential in toroidal coordinates [2]. The main equation of magnetostatics has the form

$$\nabla \times \nabla \times \vec{A} = \vec{j} \,. \tag{2}$$

Since majority of physicists are not familiar with the curl operation in this coordinate system, it is necessary to derive this equation from the scratch. It is easier to do this in terms of exterior differential forms [9] and hence to use this mathematical toolkit from the very beginning. Therefore we represent equation (2) as follows. The vector potential is replaced with co-vector $\alpha = A_i dx^i$, magnetic strength appears as 2-form of its exterior derivative $H = d\alpha$ and the master equation takes the form

$$d^* d\alpha = I \,, \tag{3}$$

where the current density I also appears as 2-form. In a coordinate system $\{u,v,\varphi\}$ with azimuthal angle φ as one of coordinates, the field in question has single component $A_\varphi(u,v)$ which is the only function to be found. Hence the equation (3) reduces to a partial differential equation and the next task is to separate variables so that it turns into one of two ordinary differential equation which can be solved as usual. Let $\{u,v,\varphi\}$ be a coordinate system with φ being azimuthal angle about a straight line which serves as the axis of symmetry and the Lame coefficients of this systems are h_u, h_v and ρ correspondingly (ρ stands for the distance from the axis of symmetry which always serves as this coefficient). Then, $v^1 = h_u du$, $v^2 = h_v dv$ and $v^3 = \rho d\varphi$ constitute a field of orthonormal frames in the space. So, if the coordinate system under consideration is toroidal one, a field of orthonormal co-vector frames is presented by

$$v^1 = \frac{a du}{\cosh u - \cos v}, \qquad v^2 = \frac{a dv}{\cosh u - \cos v}, \qquad v^3 = \frac{a \sinh u d\varphi}{\cosh u - \cos v}. \tag{4}$$

Such a frame is needed to facilitate the asterisk operation, which for 2-forms is defined by

$$^*(v^a \wedge v^b) = \varepsilon_c^{ab} v^c . \tag{5}$$

Consider circular current loop, which coincides with the focal circle in toroidal coordinates. The co-vector potential of this source is strictly azimuthal, hence, is of the form

$$\alpha = A(u,v) d\varphi . \tag{6}$$

Now we derive the equation (3) for this particular co-vector. The exterior derivative is

$$d\alpha = A_v dv \wedge d\varphi - A_u d\varphi \wedge du = \frac{(\cosh u - \cos v)^2}{a^2 \sinh u}(A_v v^2 \wedge dv^3 - A_u v^3 \wedge v^1) ,$$

its asterisk conjugate obtained with use of the orthonormal frame (4, 5) is

$$^*d\alpha = \frac{(\cosh u - \cos v)^2}{a^2 \sinh u}(A_v v^1 - A_u v^2) = \frac{\cosh u - \cos v}{a \sinh u}(A_v du - A_u dv) , \tag{7}$$

and finally, we obtain the left-hand side of the equation (3) as

$$d^* d\alpha = -\frac{1}{a^2}\left[\frac{\partial}{\partial u}\left(\frac{\cosh u - \cos v}{a \sinh u}\frac{\partial A}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{\cosh u - \cos v}{a \sinh u}\frac{\partial A}{\partial v}\right)\right] du \wedge dv ,$$

where the expression in brackets coincides with the left-hand side of the equation (3) represented in toroidal coordinates. Thus, the equation for the φ-component of the co-vector potential in these coordinates has the form

$$\frac{\partial}{\partial u}\left(\frac{\cosh u - \cos v}{a \sinh u}\frac{\partial A}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{\cosh u - \cos v}{a \sinh u}\frac{\partial A}{\partial v}\right) = 0 . \tag{8}$$

III. VARIABLES SEPARATION

The second task is to reduce the equation (8) to ordinary differential equations. Note that the fracture appeared in this equation, is nothing but ρ^{-1}, that is inverse distance from the axis of symmetry. Therefore, the left-hand side of the equation can be transformed as follows:

$$\frac{\partial}{\partial u}\left(\frac{\cosh u - \cos v}{a \sinh u}\frac{\partial A}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{\cosh u - \cos v}{a \sinh u}\frac{\partial A}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{1}{\rho}\frac{\partial A}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\rho}\frac{\partial A}{\partial v}\right) =$$

$$= \frac{1}{\sqrt{\rho}}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\left(\frac{A}{\sqrt{\rho}}\right) - \frac{A}{\sqrt{\rho}}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\frac{1}{\sqrt{\rho}} .$$

Note that cylindric coordinates z and ρ are Cartesian ones on planes φ = const. It is well known from analysis of complex variable; conformal transformation $\{z,\rho\} \to \{u,v\}$ changes the Laplace operator as follows:

$$\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} = J\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2}\right) ,$$

where J is Jacobian of the transformation which is equal to square of the Lame coefficient h_u or h_v. In this case we have

$$\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} = \frac{a^2}{(\cosh u - \cos v)}\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2}\right) .$$

Now we can obtain the last term of the equation as follows:

$$\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)\frac{1}{\sqrt{\rho}}=\frac{a^2}{(\cosh u-\cos v)^2}\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial \rho^2}\right)\frac{1}{\sqrt{\rho}}=\frac{3}{4\sqrt{\rho}\sinh^2 u}.$$

Therefore, the equation for the function A becomes

$$\frac{1}{\sqrt{\rho}}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}-\frac{3}{4\sinh^2 u}\right)\frac{A}{\sqrt{\rho}}=0.$$

It is convenient to denote

$$\Psi=\frac{A}{\sqrt{\rho}} \tag{9}$$

and suppose from physical considerations, that the function Ψ depends only on u. This property of the function to be found follows from the fact that near the loop $u = \infty$ the magnetic strength has only the v-component. Finally, we obtain the following ordinary differential equation for this single-variable function:

$$\left(\frac{\partial^2}{\partial u^2}-\frac{3}{4\sinh^2 u}\right)\Psi=0. \tag{10}$$

The final task is to solve it.

IV. RESULTS

The equation obtained is similar to the equation, which was studied in quantum mechanics due to the modified Pöschl-Teller problem [10]. The desired exact solution of this equation is constructed below by the method used when solving this problem [11]. To obtain the desired analytical solution of the equation (10) we make the following substitution:

$$\Psi=\frac{y(u)}{\sqrt{\sinh u}} \tag{11}$$

and, first of all, calculate the first and second derivatives of the function $\Psi(u)$ represented this way:

$$\Psi'=\frac{y'}{\sinh^{1/2} u}-\frac{y\cosh u}{2\sinh^{3/2} u},$$

$$\Psi''=\frac{y''}{\sinh^{1/2} u}-\frac{y'\cosh u}{\sinh^{3/2} u}+\frac{3y\cosh^2 u}{4\sinh^{5/2} u}-\frac{y}{2\sinh^{1/2} u}.$$

Substituting this into the equation (10) yields the following equation for the function $y(u)$:

$$y''-y'\coth u+(y/4)=0.$$

This equation can be rewritten in the form

$$\sinh u\frac{d}{du}\left(\frac{1}{\sinh u}\frac{dy}{du}\right)+\frac{y}{4}=0.$$

To solve it we denote

$$\frac{1}{\sinh u}\frac{dy}{du}=f(u), \qquad \sinh u\frac{df}{du}=y(u) \tag{12}$$

and apply the operator in the left-hand side of this equation to the equation (11). This yields the following equation:

$$\frac{1}{\sinh u}\frac{d}{du}\left(\sinh u\frac{df}{du}\right)+\frac{f}{4}=0.$$

Solutions of this equation are well-known:

$$f(u)=f_{-1/2}(\cosh u)$$

where $f_{-1/2}(w)$ stands for that one of Legendre functions $P_{-1/2}(w)$ or $Q_{-1/2}(w)$ which is finite under u = 0. For the reason specified below, we select the function $P_{-1/2}(\cosh u)$. Returning to the function $y(u)$ via the equation (12) yields explicit form of this function which is

$$y(u)=\frac{d}{du}P_{-1/2}(\cosh u),$$

that gives explicit form of the desired solution of the equation (10):

$$\Psi=\frac{1}{\sqrt{\sinh u}}\frac{d}{du}P_{-1/2}(\cosh u)$$

and it remains to return to the φ-component of the vector potential **A** via the equation (9):

$$A(u,v)=\frac{\sinh u}{\cosh u-\cos v}\frac{d}{du}P_{-1/2}(\cosh u). \qquad (13)$$

Note that φ-component of the potential behaves approximately as $\sinh^2 u\approx\rho^2$ near the axis ρ = 0. This expression makes it possible to obtain explicit form of the strength of the magnetic field produced by a circular current loop. Besides, it describes lines of force of the magnetic field, which are given by $A(u,v)$=const. Indeed, since, due to the equation (7), components of the magnetic strength are

$$H_u=\frac{\cosh u-\cos v}{a\sinh u}\frac{dA}{dv}, \qquad H_v=\frac{\cosh u-\cos v}{a\sinh u}\frac{dA}{du}. \qquad (14)$$

The strength $\vec{H}$ is tangent to the lines

$$A(u,v)=\text{const} \qquad (15)$$

in the half-planes φ = const. A plot of the lines of force is given on Fig. 1. Another figure, (Fig 2), which exposes lines of force of a point-like dipole obtained from the wellknown expression, allows one to compare the lines.

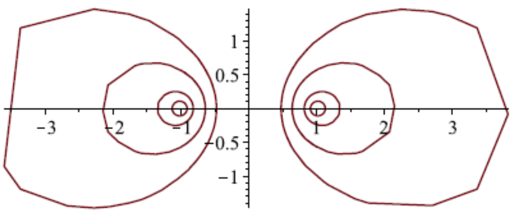

Fig. 1. Lines of force of the field of circular current loop.

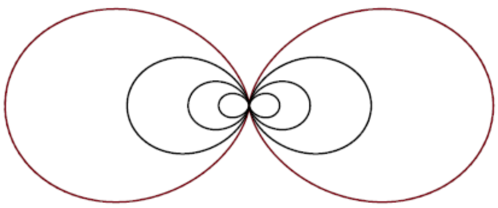

Fig. 2. Lines of force of the field of a point-like dipole.

V. DISCUSSION

Presently magnetic field of a circular current loop is known mostly in its integral form found in numerous texts on classical electrodynamics, for example, in the book [1], equation (5.36), where it is given by the expression:

$$A_{\varphi} = \frac{1}{ca}\int_0^{2\pi} \frac{\cos\varphi' d\varphi'}{(a^2 + r^2 - 2\arcsin\theta\cos\varphi')^{1/2}} . \qquad (16)$$

This expression provides rather a formal completion of the task because to obtain components of the field, one needs to make further calculations based on this form, particularly, to take a complicated integral that remained undone for many decades. Recently, calculations of the strength components have been completed in various coordinate systems [2]. The expressions obtained look not very complicated, but contain complicated substitutions that makes it difficult to use them. A need for a simpler ready-to-use form of the vector potential and strength components looks quite evident.

It must be pointed out that the subscript φ is misused in this formula. The point is that the genuine φ-component is not magnitude of the co-vector which was calculated in the equation (16). These two values are in the following relation:

$$A_{\varphi} = \rho|\alpha| , \qquad (17)$$

so, to obtain the genuine φ one needs to multiply the right-hand side by $\rho = r \sin \theta$. However, behavior of this component near the axis $\rho = 0$ is known and does not coincide with that of the result obtained. Indeed, the field is close to uniform there, and component of the potential under consideration. As for the right-hand side of the equation (16), it does not vanish on the axis and even after multiplying by $r \sin \theta$, it does not do it quadratically as it should. Consequently, the integral itself is a wrong expression of the potential.

Usually, closed form expressions come out from straightforward solutions of the field equation obtained by the method of variables separation. This method provides the complete entire linear space of solutions that is especially important in all linear theories. However, despite that is linear, no expressions obtained this way are found in the literature even for the case with zero right-hand side. In case of the field of circular current loop, the task is very similar to commonplace ones encountered in scalar physics because in this special case the vector to be found has single non-zero component and the equation (2) for it reduces to an ordinary differential one almost the same way as scalar ones. Nevertheless, the only approach applied in this case, consists in writing down the expression (16). The integral was taken in the work [8], however the integration procedure is not shown. The only component of the vector potential obtained this way is given by an expression which contains elliptic integrals of complicated expressions. Unlike all this, derivation of an exact closed from expression for the field as a straightforward solution of the master equation and, particularly, its result presented in this work, are more transparent and correct.

VI. CONCLUSION

The field of circular current loop is constructed as an analytic solution of the field equation in toroidal coordinates. Exact closed form expression for the single φ-component of the (co-)vector potential and components of the field strength are obtained. Lines of force of the field are plotted and plots of these lines are compared to that of point-like magnetic dipole obtained another way. Visual comparison shows that in the point-like limit the earlier turns into the latter. This fact signifies that the solution obtained describes the field of circular current loop properly. Thus, in spite of numerous publications on the subject, magnetic field of circular current loop is obtained in this work for the first time. The method for solving the field equation used in this work was presented before in our book [11].

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